Graph Theory Review

Niklas Bühler

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Preface

This is a summary of the most important definitions, theorems and proofs for the Graph Theory lecture at KIT. It is based on a short review done by Prof. Axenovich at the end of winter term 2019/20, which was based on her lecture notes, which themselves are based on the book Graph Theory[1]. I added a short sketch to most proofs in order to make memorizing it easier.

[1] Reinhard Diestel. *Graph theory.* Fifth edition. Graduate texts in mathematics ; 173. Berlin; [Heidelberg]: Springer, [2017]. ISBN: 9783662536216; 3662536218.

1 Basic notions

Some common proof techniques

1. Induction
2. Extremal principle with contradiction: *Consider a longest path/largest matching/...*
3. Counting arguments: *Double counting, Pigeonhole principle, Parity arguments*
4. Algorithmic, iterative approach: *Just do it!*
5. Ramsey: *Either the red coloring has a structure we want or if not then that implies some structural information in the blue coloring.*
6. Probabilistic method: $\mathbb{P}(\bigcup \textit{Bad event}) < 1$, *therefore some object with good properties exists.*
   Compute $\mathbb{E}X$*, using linearity of* $\mathbb{E}$*.*
   Alterations: random object has some unwanted structure, simply destroy it by removing an edge, etc.
7. Apply a theorem!

Theorem 1 (Tree equivalence theorem)**.** The following statements are equivalent:

1. G is a tree, i.e. connected and acyclic.
2. G is minimally connected.
3. G is maximally acyclic.
4. G is 1-degenerate.
5. G is connected and $|E| = |V| - 1$.
6. G is acyclic and $|E| = |V| - 1$.
7. G is connected and every non-trivial subgraph has a vertex v: $d(v) \leq 1$.
8. Any two vertices of G are joined by a unique path.

Remark 2 (Characterization of bipartite graphs)**.** *G is bipartite $\iff$ G has no odd cycle.*

Proof. As G is bipartite, every cycle has to be even. Consider a partitioning into sets A and B by distances to a vertex v modulo 2. Then, for every edge ab look at shortest a-v-path and b-v-path and show that a and b can't be in the same partition.

Assume $G = A\dot{\cup}B$ bipartite. Then any cycle has the form $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1$, so even length.

Assume G has no cycles of odd length and is connected, otherwise treat components separately.
Let $v \in V$, $A = \{u \in V | \operatorname{dist}(u, v) \equiv 0 \pmod 2\}, B = \{u \in V | \operatorname{dist}(u, v) \equiv 1 \pmod 2\}$.
A and B are independent sets: Let $u_1u_2 \in E, P_1$ a shortest $u_1 - v$-path, P_2 a shortest $u_2 - v$-path. Then $W := P_1 \cup P_2 \cup \{u_1u_2\}$ is a closed walk. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$, then W is a closed odd walk, thus G contains an odd cycle, a contradiction. Thus, $\forall u_1u_2$, u_1 and u_2 are in different parts A or B. □

Definition 3. An Euler tour is a walk that visits every edge exactly once.

Theorem 4 (Euler tours)**.** A connected graph has an Euler tour $\iff$ every vertex has even degree.

Proof. Use extremal principle with contradiction: Consider a longest walk W with non-repeating edges. Then show that it has to be closed and contain all edges, otherwise W was not maximal.

A connected graph has an Euler tour $\iff$ every vertex has even degree.
Assume G is connected and has an Euler tour. Then by definition of the tour, there is an even number of edges incident to each vertex.

Assume G is connected with all vertices of even degree. Consider a walk $W := v_0, e_0, \ldots, v_k$ with non-repeated edges and having largest possible number of edges.
First, W has to be a closed walk: If $v_0 \neq v_k$, v_0 is incident to an odd number of edges in W, a contradiction to W's maximality.

Also, W contains all the edges of G: Otherwise, by G's connectivity, there is an edge $e = v_i x$ of G that is incident to v_i and not contained in W. Then the walk $x, e, v_i, e_i, v_{i+1}, \ldots, v_k, e_0, v_1, e_1, \ldots, v_i$ is longer than W, a contradiction. Therefore, W is a closed walk containing all edges of G, an Euler tour. □

2 Matchings

Theorem 5 (Hall's marriage theorem)**.** *G bipartite with sets A, B. G has a matching saturating $A \iff |N(S)| \geq |S| \; \forall S \subseteq A$.*

Proof. Do induction on $|A|$. Consider two cases:
Case 1: $|N(S)| \geq |S|+1 \; \forall S \subsetneq A$: Simply take out one edge and it's vertices, get a matching by induction hypothesis and add the edge to that matching.
Case 2: $\exists A' \neq \emptyset$, such that $|N(A')| = |A'|$: Consider $G' := G[A' \cup N(A')]$. Again, get a matching by induction hypothesis and combine that with a matching in $G - G'$, also by induction hypothesis.

Induction on $|A|$:
For $|A| = 1$, the assertion is true. Let $|A| \geq 2$:

Case 1. $|N(S)| \geq |S| + 1 \; \forall S \subsetneq A$.
Pick an edge ab ($a \in A, b \in B$) and consider $G' := G - \{a, b\}$.
Every set $\emptyset \neq S \subseteq A \setminus \{a\}$ satisfies

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|,$$

so by induction hypothesis G' contains a matching of $A \setminus \{a\}$, so together with the edge ab, this is a matching of A.

Case 2. $\exists A' \neq \emptyset$ with $B' := N(A')$ and $|B'| = |A'|$.
By induction hypothesis, $G' := G[A' \cup B']$ contains a matching of A'. But $G - G'$ also satisfies the marriage condition: $\forall S \subseteq A \setminus A'$ with $|N_{G-G'}(S)| < |S|$ we would have $|N_G(S \cup A')| < |S \cup A'|$, contrary to our assumption.
Again, by induction, $G - G'$ contains a matching of $A \setminus A'$.
These two matchings result in a matching saturating A. □

Theorem 6 (Kőnig's theorem)**.** *If G is bipartite, then the size of a largest matching is the same as the size of a smallest vertex cover.*

Proof. *A vertex cover contains at least one vertex of every edge of a matching, so $m \leq c$.*
Define $U' = \{b : \text{an alternating path ends in } b\}$ and $U = U' \cup \{a : ab \in E(M), b \notin U'\}$. U is a vertex cover and $|U| = m$.

Let $G = A \dot\cup B$ and let c be the size of a smallest vertex cover and m the size of a largest matching. Since a vertex cover contains at least one vertex from every matching edge, $c \geq m$. To show $m \geq c$ consider a largest matching M and let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\},$$

$$U = U' \cup \{a : ab \in E(M), b \notin U'\}.$$

Note that $|U| = m$. U is a vertex cover, i.e. every edge of G contains a vertex from U: If $ab \in E(M)$, then either a or b is in U. For $ab \notin E(M)$:

Case 0. $a \in U$. Done.

Case 1. a is not incident to M. Then ab is an alternating path. b has to be incident to M, otherwise $M \cup \{ab\}$ is a larger matching, a contradiction.

Case 2. a is incident to M. Then $ab' \in M$ for some b'. Since $a \notin U$, $b' \in U$, thus there is an alternating path P ending in b'. If P contains b, then $b \in U$, otherwise $Pb'ab$ is an alternating path ending in b, so $b \in U$. □

Theorem 7 (Tutte's theorem)**.** G has a perfect matching $\iff$ $\forall S \subseteq V$ $q(G - S) \leq |S|$.
For a graph G, $q(G)$ denotes the amount of odd components of G.

3 Connectivity

Theorem 8 (Menger's theorem)**.** The maximum number of A-B-paths in G is equal to the minimum number of vertices separating A from B.

Theorem 9 (Global version of Menger's theorem)**.** G is k-connected $\iff$ $\forall a, b \in V$ there are k independent a-b-paths.

Theorem 10 (Ear decomposition)**.** G is 2-connected $\iff$ G has an ear decomposition starting from any cycle in G.

Proof. *Do induction over a given ear decomposition to show that it is 2-connected. For the other implication, take a maximal subgraph obtained by an ear decomposition starting from a cycle C in G and show that it is induced and equal to G, both times contradicting its maximality if not.*

Assume there is such an ear decomposition starting from C:

$$C = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k = G$$

Do induction on i:
$G_0 = C$ is clearly 2-connected. If G_{i+1} contains a cut-vertex, it must be on the added ear. But deleting a vertex from the ear does not disconnect G_{i+1} since an ear is contained in a cycle.

Assume G is 2-connected and C is a cycle in G. Let $H =$ largest subgraph obtained by ear decomposition starting with C. H is induced subgraph of G, otherwise an edge with two vertices in $V(H)$ is an ear, contradicting H's maximality.
Assume $H \neq G$. As G is connected, there is an edge $e = uv, u \in V(H), v \notin V(H)$. Since $G - u$ is connected, consider a $v - w$-path P in $G - u$ for some $w \in V(H) - u$. Let w' be the first vertex from $V(H) - u$ on P. Then $Pw' \cup uv$ is an ear of H, contradicting its maximality. □

Definition 11 (Block)**.** A maximal connected subgraph of G without a cut vertex is called a *block* of G.

Remark 12 (Blocks)**.** *B is a block of G $\iff$ B is a bridge or a maximal 2-connected subgraph of G.*

4 Planarity

Theorem 13 (Euler's formula)**.**

$$n - m + f = 2,$$

where $n = |G|$, $m = ||G|| = |E(G)|$ and f is the number of faces of G.

Proof. *Fix n and do induction on m. If $m \leq n - 1$, the graph is a tree. Otherwise, consider $G' := G - e$ for an edge e that is contained in a cycle.*

Note that e lies on the boundary of exactly two faces. Remove e and apply induction hypothesis.

Fix n and do induction on m.
For $m \leq n-1$, G is a tree and because $m = n-1$, we have $n-(n-1)+f = 1+1 = 2$.
So let $m \geq n$. Then G has an edge e in a cycle.
Let $G' := G - e$. Then e lies on the boundary of exactly two faces, f_1, f_2.
One can show that $F(G') = F(G) - \{f_1, f_2\} \cup \{f'\}$, where $f' = f_1 \cup f_2 \setminus e$.
Let n', m', f' be the number of vertices, edges and faces in G'. Then we see that $n = n', m = m'+1, f = f'+1$. So, $n - m + f = n' - m' + f' = 2$. □

Definition 14 (Minor)**.** X is a *minor* of G ($X \preceq G, MX \subseteq G$), if X can be obtained from G by successive vertex deletions, edge deletions and edge contractions.

Definition 15 (Topological minor)**.** G is a *single-edge subdivision* of X, if $V(G) = V(X) \cup \{v\}$ and $E(G) = E(X) - xy + xv + vy$ for $xy \in E(X)$ and $v \notin V(X)$.
G is a *subdivision* of X, if it can be obtained from X by a series of single-edge subdivisions.
X is a *topological minor* of G ($TX \subseteq G$), if a subgraph of G is a subdivision of X.

Theorem 16 (Kuratowski's theorem)**.** G is planar $\iff G \not\supseteq TK_5, TK_{3,3} \iff G \not\supseteq MK_5, MK_{3,3}$.

Definition 17 (Dual graph)**.** The dual graph of a plane graph G has a vertex for every face of G. It has an edge, wherever two faces of G are separated by an edge (loops if the same face appears on both sides of an edge).

Theorem 18 (5-Color theorem)**.** $\forall G$ planar: $\chi(G) \leq 5$.

Proof. Do induction on $|G|$.
Assume $|G| > 5$ *and* G *maximally planar, i.e. plane triangulation. Then by Euler's formula* $\exists v : d(v) \leq 5$.
By induction there is a coloring c *of* $G - v$ *using 5 colors. Assume* c *assigns 5 colors to* $N(v) = \{v_1, \ldots, v_5\}$*, in clockwise order, and* $c(v_i) = i$.

If v_1, v_3 or v_2, v_4 are not linked by paths of colors only 1 and 3 or only 2 and 4, then v_1 can be colored in 3 or v_2 can be colored in 4. So assume there is a v_1-v_3-path only colored 1 and 3 and a v_2-v_4-path only colored 2 and 4. But then these paths must cross, a contradiction to the planarity of G.

Do induction on $|G|$.
If $|V(G)| \leq 5$, the result is trivial.
Assume $|G| > 5$ and G is maximally planar, i.e. has a plane embedding that is a triangulation. Then by Euler's formula $\exists v : d(v) \leq 5$.
By induction there is a coloring c of $G - v$ using 5 colors. Assume c assigns 5 colors to $N(v) = \{v_1, \ldots, v_5\}$, in clockwise order, and $c(v_i) = i$.

Consider a subgraph induced by all vertices colored 1 or 3:
v_1 and v_3 are in different components of that subgraph, we can switch colors 1 and 3 in the component of v_1 and color v in 1. So assume v_1 and v_3 are in the same component and there is a path connecting them, colored in 1 and 3 only.

Now consider a subgraph induced by all vertices colored 2 or 4:
If v_2 and v_4 are in different components of that subgraph, we can switch colors 2 and 4 in the component of v_2 and color v in 2. So assume not, then there is a path connecting them, colored in 2 and 4 only.

But this means, these two paths cross each other, contradicting the planarity of G. □

Theorem 19 (5-List-Color theorem)**.** $\forall G$ planar: $\chi_l(G) \leq 5$.

Proof. Prove a stronger statement:
Let G be an outer triangulation (max. planar) with two adjacent vertices x, y on the other triangle. Let $L : V(G) \to 2^{\mathbb{N}}$ be a list assignment, such that $|L(x)| = |L(y)| = 1, L(x) \neq L(y), |L(z)| = 3$ for any other vertex z on the outer face and $|L(z)| = 5$ for every vertex not on the bounded face.
Then G is L-colorable.

Do induction on $|G|$ with an obvious basis for $|G| = 3$. Consider an outer triangulation G on more than 3 vertices.

Case 1. There is a chord uv.
Let $G = G_1 \cup G_2$, such that $\{u, v\} = V(G_1) \cap V(G_2)$, $|G| > |G_i| \geq 3$, G_i

is an outer triangulation. W.l.o.g. x, y are on the outer face of G_1. Apply induction to G_1 and obtain a proper L-coloring c' of G_1. Then apply induction on G_2 with u, v taking the roles of x, y and list assignments L' such that $L'(u) = \{c'(u)\}, L'(v) = \{c'(v)\}, L'(z) = L(z)$ for $z \notin \{x, y\}$. Then there is a proper L'-coloring c'' of G_2. These colorings coincide on u and v, so together they form a proper coloring c of G, i.e. $c(v) = c'(v)$ for $v \in V(G_1)$ and $c(v) = c''(v)$ for $v \in V(G_2)$.

Case 2. There is no chord.
Let z be a neighbor of x on the boundary of the outer face, $z \neq y$. Let Z be the set of neighbors of z not on the outer face. Let $L(x) = \{a\}, L(y) = \{b\}$. Let $c, d \in L(z)$ such that $c \neq a$ and $d \neq a$. Let $G' = G - z$. Finally, let L' be the list assignment for $V(G')$ such that $L'(v) = L(v) - \{c, d\}$ for $v \in Z$ and $L'(v) = L(v)$ for $v \notin Z$.
By induction, G' has a proper L'-coloring c'. Extend c' to a coloring c of G: Let $c(v) = c'(v)$ if $v \neq z$. Let $c(z) \in \{c, d\} \setminus \{c'(q)\}$ where q is the neighbor of z on the outer face, $q \neq x$. z then has a color different from each of its neighbors, so c is a proper L-coloring.

□

5 Colorings

Theorem 20 (Brook's theorem)**.** Let G be a connected graph.
Then $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle.

Proof. Do induction on n. If G has a cut-vertex v, apply induction on $G_1 \cup G_2 = G$*, where* $\{v\} = V(G_1) \cap V(G_2)$*,* $|G_1|, |G_2| < |G|$*. This gives* $\chi(G_i) \leq \Delta(G)$ *and the colors can be permutated so that v has the same color in both colorings, resulting in a combined coloring of G.*
If G has no cut-vertex and $\Delta(G) \geq 3$*, then G is 2-connected.*
Case 1. $\exists v : d(v) \leq \Delta - 1$.
Order the vertices $v_i, \ldots, v_n$*, such that* $v = v_n$ *and each* v_i *has a neighbor with larger index and color G greedily. At step i, there are at most* $\Delta - 1$ *neighbors of* v_i *colored, so there is an avaiable color for* v_i*.*
Case 2. $\forall v : d(v) = \Delta$.
Consider $x, y, z \in V$*, s.t.* $xy \notin E$*,* $xv, yv \in E$ *and* $G - \{x, y\}$ *is connected. Order the vertices* $v_i, \ldots, v_n$*, such that* $x = v_1, y = v_2, v = v_n$ *and each* v_i *has a neighbor with larger index and color G greedily.*

x and y get the same color and as in the first case, v_i can be colored. v_n has Δ colored neighbors, but $c(x) = c(y)$.

Induction on n. Assume $|G| > 3$.
If G has a cut-vertex v, apply induction on G_1, G_2, s.t. $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ and $|G_1| < |G|$ and $|G_2| < G$.
If each of G_1, G_2 is not complete or an odd cycle, then $\chi(G_i) \leq \Delta(G_i) \leq \Delta(G)$.
If G_i is complete or an odd cycle, $\Delta(G_i) < \Delta(G)$ and $\chi(G_i) = \Delta(G_i) + 1 \leq \Delta(G)$. By making sure that the color of v is the same in an optimal proper coloring of G_1 and G_2 we see that $\chi(G) \leq \Delta(G)$.

Note also that if $\Delta(G) \leq 2$, the theorem holds trivially, so we assume $\Delta(G) \geq 3$. Then G is 2-connected.

Case 1. $\exists v : d(v) \leq \Delta - 1$.
Order the vertices of G $v_1, \ldots, v_n$, such that $v = v_n$ and each v_i has a neighbor with larger index. Color G greedily in this ordering.
At step i, there are at most $\Delta - 1$ neighbors of v_i colored, so there is an available color for v_i.

Case 2. $\forall v : d(v) = \Delta$.
Consider vertices x, y, z, s.t. $xy \notin E$ and $xv, yv \in E$ and $G - \{x, y\}$ is connected. Order the vertices of G $v_1, \ldots, v_n$, such that $x = v_1$, $y = v_2$, $v = v_n$ and each v_i $(3 \leq i < n)$ has a neighbor with larger index. Color G greedily in this ordering.
v_1 and v_2 get the same color and as in the previous case, v_i has at most $\Delta - 1$ colored neighbors $(3 \leq i < n)$, so it can be colored in the remaining color. At the last step, v_n has Δ colored neighbors, but two of them, v_1, v_2 have the same color, so there are at most $\Delta = 1$ colors used by neighbors of v_n. Thus v_n can be colored in the remaining color. □

Lemma 21 (Greedy coloring)**.** $\chi(G) \leq \Delta(G) + 1$

Proof. For any connected graph G and any vertex v there is an ordering of the vertices of G: $v_1, \ldots, v_n$, such that $v = v_n$ and $\forall 1 \leq i < n$ v_i has a higher indexed neighbor:

Consider a spanning tree T of G and create a sequence of sets $X_1, \dots, X_{n-1}$ with $X_1 = V, X_i = X_{i-1} - \{v_{i-1}\}$, where v_i is a leaf of $T[X_i]$ not equal to v. Then $v_1, \dots, v_n$ is a desired ordering. □

Lemma 22 (Clique number and chromatic number)**.** $\omega(G) \leq \chi(G)$.

Definition 23 (Perfect graphs)**.** G is perfect $\iff \omega(H) = \chi(H)\ \forall H \subseteq G$, induced.

Theorem 24 (Weak perfect graph theorem)**.** G is perfect $\iff \overline{G}$ is perfect.

Theorem 25 (Strong perfect graph theorem)**.** G is perfect $\iff$ G has no odd hole (odd cycle on at least 5 vertices) or antihole (complement of an odd hole) as induced subgraph.

Theorem 26 (Vizing's theorem)**.** $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$

Theorem 27 (Kőnig's theorem)**.** If G is bipartite, then $\chi'(G) = \Delta(G)$.

Proof. *$\chi'(G) \geq \Delta(G)$, because the edges incident to a vertex of maximum degree require distinct colors.*
For $\chi'(G) \leq \Delta(G)$ do induction on $||G||$, with trivial base. Let $e = yt \in E$, then by induction c is a proper edge coloring of $G' = G - e$ using colors from $\{1, \dots, \Delta(G)\}$. As $d_{G'}(x), d_{G'}(y) \leq \Delta(G) - 1$, there are color sets $\emptyset \neq Mis(x), Mis(y) \subseteq [\Delta(G)]$, s.t. no edge incident to v uses colors from $Mis(v)$. Consider two cases:
Case 1: $Mis(x) \cap Mis(y) \neq \emptyset$: Let $c(e) \in Mis(x) \cap Mis(y)$.
Case 2: $Mis(x) \cap Mis(y) = \emptyset$: Let $\alpha \in Mis(x), \beta \in Mis(y)$. Consider a longest path P colored α and β starting at x. y is not a vertex in P, as it is not incident to β, and not the other endpoint of P because of parity. Switch colors α and β on P. Then we obtain a proper edge-coloring in which $\beta \in Mis(x) \cup Mis(y)$, which allows e to be colored β.

$\chi'(G) \geq \Delta(G)$, because the edges incident to a vertex of maximum degree require distinct colors.

For the upper bound, $\chi'(G) \leq \Delta(G)+1$, do induction on $||G||$. Base $||G|| = 1$ is trivial. Let G be a graph, $||G|| > 1$, assume that the assertion holds for all graphs with less edges.
Let $e = yt \in E$. By induction there is a proper edge coloring c of $G' = G - e$ using colors from $\{1, \dots, \Delta(G)\}$.

In G' both x and y are incident to at most $\Delta(G) - 1$ edges. Thus there are color sets $\emptyset \neq \text{Mis}(x), \text{Mis}(y) \subseteq [\Delta(G)]$, where $\text{Mis}(v)$ is the set of "missing" colors, i.e. colors not used on edges incident to v.

Case 1. $\text{Mis}(x) \cap \text{Mis}(y) \neq \emptyset$: Let $\alpha \in \text{Mis}(x) \cap \text{Mis}(y)$, color e with α. This gives $\chi'(G) \leq \Delta(G)$.

Case 2. $\text{Mis}(x) \cap \text{Mis}(y) = \emptyset$: Let $\alpha \in \text{Mis}(x)$ and $\beta \in \text{Mis}(y)$. Consider a longest path P colored α and β starting at x. Because of parity, P does not end in y, and because y is not incident to β, y is not a vertex on P. Switch colors α and β on P. Then we obtain a proper edge-coloring in which $\beta \in \text{Mis}(x) \cup \text{Mis}(y)$, which allows e to be colored β. Thus $\chi'(G) \leq \Delta(G)$. □

Definition 28 (List-colorable, List-chromatic-number)**.** Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$.

G is L-list-colorable if there is a coloring $c : V \to \mathbb{N}$ such that $c(v) \in L(v)$ $\forall v \in V$ and adjacent vertices have different colors.

G is k-list-colorable, if G is L-list-colorable for every L with $L(v) = k$ for every $v \in V$.

$\chi_l(G)$ is the smallest k such that G is k-list-colorable.

6 Flows

Theorem 29 (Ford-Fulkerson theorem)**.** Let $N = (G, s, t, c)$ be a network. Then

$$\max\{|f| : f \text{ is an N-flow}\} = \min\{c(S, \overline{S}) : (S, \overline{S}) \text{ is a cut}\}.$$

Also, there is an integral flow $f : T \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

7 Substructures in dense graphs

Definition 30. Extremal number The extremal number $\text{ex}(n, H)$ is defined as $\max\{||G|| : |G| = n, G \not\supseteq H\}$.

$\mathrm{EX}(n, H) := \{G : ||G|| = \mathrm{ex}(n, H), |G| = n, G \not\supseteq H\}$ is the set of H-free graphs on n vertices with $\mathrm{ex}(n, H)$ edges.

Definition 31 (Turán graph)**.** The *Turán graph* $T(n, r)$ is the unique complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}.
Notation: $t(n, r) = ||T(n, r)||$. If $n = r * s$, $T(n, r)$ is also denoted by K_r^s.

Theorem 32 (Turán's theorem)**.** Any graph G with n vertices, $\mathrm{ex}(n, K_r)$ edges and $K_r \not\subseteq G$ is a $T_{r-1}(n)$.
In other words, $\mathrm{EX}(n, K_r) = \{T(n, r-1)\}$.

Remark 33 (Binomial coefficient)**.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 34 (Erdős-Stone-Simonovits)**.** For any graph H and for any fixed $\epsilon > 0$, there is n_0 such that for any $n \geq n_0$,

$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right)\binom{n}{2} \leq \mathrm{ex}(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right)\binom{n}{2}.$$

Definition 35 (ϵ-regularity)**.** Let $||X, Y||$ denote the number of edges between X and Y. Then the *density* $d(X, Y)$ between X, Y is defined as $d(X, Y) = \frac{||X,Y||}{|X||Y|}$.

For $\epsilon > 0$, the pair (X, Y) is *ϵ-regular*, if $|d(X, Y) - d(A, B)| \leq \epsilon$ for all $A \subseteq X, B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$.

An *ϵ-regular partition* of G is a partition $V = V_0 \dot\cup \dots \dot\cup V_k$ such that:

1. $|V_0| \leq \epsilon|V|$,
2. $|V_1| = |V_2| = \cdots = |V_k|$,
3. All but at most ϵk^2 of the pairs (V_i, V_j) are ϵ-regular.

8 Substructures in sparse graphs

Conjecture 36 (Hadwiger's conjecture)**.** $\chi(H) = r \Rightarrow H \supseteq MK_r$

9 Ramsey theory

Definition 37 (Ramsey number)**.** The *Ramsey number* $R(k)$ is the smallest $n \in \mathbb{N}$, such that every 2-edge-coloring of K_n contains a monochromatic K_k.

The *asymetric Ramsey number* $R(k, l)$ is the smallest $n \in \mathbb{N}$, such that every red-blue edge-coloring of K_n contains a red K_k or a blue K_l.

The *graph Ramsey number* $R(G, H)$ is the smallest $n \in \mathbb{N}$, such that every red-blue edge-coloring of K_n contains a red G or a blue H.

The *hypergraph Ramsey number* $R_r(l_1, \ldots, l_k)$ is the smallest $n \in \mathbb{N}$, such that for every k-coloring of $\binom{[n]}{r}$, there is an $i \in [k]$ and a $V \subseteq [n]$ with $|V| = l_i$, such that all sets in $\binom{V}{r}$ have color i.

Remark 38 (On proving graph Ramsey numbers)**.** *For the lower bound, construct a coloring that doesn't contain the red or blue subgraph.*
For the upper bound, given a coloring, show that either the blue or the red subgraph can be found.

Theorem 39 (Ramsey)**.** For any $k \in \mathbb{R}$ we have $\sqrt{2}^k \leq R(k) \leq 4^k$.

Proof. For the lower bound, use the probabilistic method, by constructing a coloring of $K_{2^{k/2}}$. Then show that the probability of a monochromatic k-clique is less than 1.
For the upper bound, consider an edge-coloring of $G = K_{4^k}$ in red and blue. Let x_1 be an arbitrary vertex and $X_1 = V$. Then let X_{i+1} be the largest monochromatic neighborhood of x_i in X_i and call its color c_i. Then let x_{i+1} an arbitrary vertex of X_{i+1}. Note that $|X_{i+1}| \geq \lceil \frac{|X_i - 1|}{2} \rceil \geq 4^k/2^i$. Thus $|X_i| > 0$ as long as $i \leq 2k$. Of $c_1, \ldots, c_{2k-1}$, at least k are the same by pigeonhole principle. The vertices belonging to that color induce a monochromatic k-vertex clique.

For the upper bound, consider an edge-coloring of $G = K_{4^k}$ in red and blue. Construct a sequence of vertices $x_1, \ldots, x_{2k}$, a sequence of vertex sets $X_1, \ldots, X_{2k}$ and a sequence of colors $c_1, \ldots, c_{2k-1}$ as follows:
Let x_1 be an arbitrary vertex and $X_1 = V(G)$.
Let X_{i+1} be the largest monochromatic neighborhood of x_i in X_i, call this

color c_i. Let x_{i+1} be an arbitrary vertex in X_{i+1}.

We see that $|X_{i+1}| \geq \lceil \frac{|X_i - 1|}{2} \rceil \geq 4^k/2^i$. Thus $|X_i| > 0$ as long as $2k > (i-1)$, i.e. as long as $i \leq 2k$. Consider vertices $x_1, \ldots, x_{2k}$ and colors $c_1, \ldots, c_{2k-1}$. At least k of these colors, say $c_{i1}, \ldots, c_{ik}$ are the same by pigeonhole principle, say red. Then $x_{i1}, \ldots, x_{ik}$ induce a k-vertex clique with all edges being red.

For the lower bound, construct a coloring of K_n, $n = 2^{k/2}$ with no monochromatic clique of size k. Color each edge red with probability $\frac{1}{2}$, otherwise blue. Let S be a fixed set of k vertices. Then

$$\text{Prob}(S \text{ induces a red clique}) = 2^{-\binom{k}{2}}.$$

So Prob(S induces a monochromatic clique) $= 2^{-\binom{k}{2}+1}$. Thus

$$\text{Prob}(\exists \text{ monochromatic clique on } k \text{ vertices}) \leq \binom{n}{k} 2^{-\binom{k}{2}+1}$$

$$\leq \frac{n^k}{k!} 2^{-k^2/2+k/2+1} \leq \frac{2^{k/2+1}}{k!} < 1.$$

□

10 Hamiltonian Cycles

Definition 40 (Hamiltonian cycle). A Hamiltonian cycle is a cycle that visits every vertex exactly once.

Theorem 41 (Dirac's theorem). If $|G| =: n \geq 3$ and $\delta(G) \geq n/2$, then G has a Hamiltonian cycle.

Proof. *G is connected, as $\delta \geq n/2$. Consider a longest path $P = (v_0, \ldots, v_k)$ and note that $N(v_0), N(v_k) \subseteq V(P)$. Show by pigeonhole principle that there is a cycle C on $k+1$ vertices. If $k+1 = n$, C is a Hamiltonian cycle. If not, then there is a vertex $v \notin V(C)$ that is adjacent to a vertex of C as G is connected. Then v and C induce a path on $k+2$ vertices, contradicting the maximality of P.*

G is connected, otherwise the smallest component has vertices of degree at most $n/2-1$, a contradiction.
If $P=(v_0,\dots,v_k)$ is a longest path, then $N(v_0),N(v_k)\subseteq V(P)$.

There is a cycle C on $k+1$ vertices in G:
Either by pigeonhole principle $v_0v_k\in E(G)$, as $|N(v_0)|,|N(v_k)|\geq n/2$ and $k\leq n-1$, or $\exists i$ such that $v_0v_{i+1},v_iv_k\in E(G)$.

If $k+1=n$, C is a Hamiltonian cycle and we are done.
If $k+1<n$, since G is connected, there is a vertex $v\notin V(C)$ adjacent to a vertex in C. Then v and C induce a graph that contains a spanning path, i.e. a path on $k+2$ vertices, a contradiction to the maximality of P. □

11 Random graphs

Definition 42 (Erdős-Rényi model of random graphs)**.** $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p\in[0,1]$.

A *property* $\mathcal{P}$ is a set of graphs, e.g. $\mathcal{P}=\{G : G \text{ is } k\text{-connected}\}$.

Let $(p_n)\in[0,1]^{\mathbb{N}}$ be a sequence. We say that $G\in\mathcal{G}(n,p_n)$ *almost always* has property $\mathcal{P}$ if $\mathrm{Prob}(G\in\mathcal{G}(n,p_n)\cap\mathcal{P})\to 1$ for $n\to\infty$. If furthermore (p_n) is constant p, we also say that *almost all* graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}$.

A function $f:\mathbb{N}\to[0,1]$ is a *threshold function* for a property P if:

- For all $(p_n)\in[0,1]^{\mathbb{N}}$ with $p_n/f(n)\xrightarrow{n\to\infty}0$ the graph $G\in\mathcal{G}(n,p_n)$ almost always does **not** have property $\mathcal{P}$.
- For all $(p_n)\in[0,1]^{\mathbb{N}}$ with $p_n/f(n)\xrightarrow{n\to\infty}\infty$ the graph $G\in\mathcal{G}(n,p_n)$ almost always does have property $\mathcal{P}$.

Not all properties have a threshold function.

Lemma 43. Let $G\in\mathcal{G}(n,p), S\subseteq V(G)$ and H a fixed graph on m edges and vertex set S. Then

$$\mathrm{Prob}(G[S]=H)=p^m(1-p)^{\binom{|S|}{2}-m} \text{ and } \mathrm{Prob}(H\subseteq G[S])=p^m.$$

Lemma 44. Let $G \in \mathcal{G}(n,p)$, let H be a fixed graph. Then

$$\text{Prob}(H \underset{ind}{\subseteq} G) \xrightarrow{n\to\infty} 1.$$

Lemma 45. Let $n \geq k \geq 2, G \in \mathcal{G}(n,p)$. Then

$$\text{Prob}(\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}} \text{ and } \text{Prob}(\omega(G) \geq k) \leq \binom{n}{k}p^{\binom{k}{2}}.$$

Theorem 46 (Erdős)**.** For any $k \geq 2$ there is a graph G on $\sqrt{2}^k$ vertices such that $\alpha(G) < k$ and $\omega(G) < k$. This implies $R(k,k) \geq 2^{k/2}$.

Proof. Let $n = \sqrt{2}^k$ and consider $G \in \mathcal{G}(n, 1/2)$. Then

$$\mathbb{P}((\alpha(G) \geq k) \vee (\omega(G) \geq k)) \leq \mathbb{P}(\alpha(G) \geq k) + \mathbb{P}(\omega(G) \geq k) \leq 2^{-\binom{k}{2}+1} < 1.$$

Thus $\mathbb{P}((\alpha(G) < k) \wedge (\omega(G) < k)) > 0$, so there is a graph G such that $\alpha(G) < k$ and $\omega(G) < k$. □

Theorem 47 (Erdős-Hajnal)**.** For any integer $k \geq 3$ there is a graph G with $\text{girth}(G) > k$ and $\chi(G) > k$.