Cooking With Curry

λ-poetry

Niklas Bühler

2020

Contents

Preface

The lambda calculus is a formal system that is used to express computations. Like the Turing machine, it's a universal model of computation, however it's much simpler and more elegant than those bulky machines.

The following pages are filled with fundamental datastructures and the most important functions operating on them, in the untyped lambda calculus.

The datastructures presented differ from ordinary, imperative datastructures, as they are purely functional. That means they don't describe where and how the data is stored, but rather how functions are applied to that data.

Variable names are often chosen as a hint on the value they're holding, but aren't elaborated on. Their exact purpose and meaning is left open for exploration.

Have fun! – August 2020

Chapter 1

Introduction

There are three main constructs in the untyped lambda calculus:

1. **Variables** x,
2. **Abstractions** $\lambda \mathrm{x}\,.\,\mathrm{t}$ and
3. **Applications** $\mathrm{f}\ \ \mathrm{x}$.

$$\mathbf{id} = \lambda \mathrm{x} \,.\, \mathrm{x}$$

Chapter 2

Booleans

$$\textbf{True} = \lambda \mathrm{t}\,.\,\lambda \mathrm{f}.\ \ \mathrm{t}$$

$$\textbf{False} = \lambda \mathrm{t}\,.\,\lambda \mathrm{f}.\ \ \mathrm{f}$$

$$\textbf{Not} = \lambda \mathrm{b}.\ \ \mathrm{b}\ \ \textbf{False}\ \ \textbf{True}$$

$$\textbf{And} = \lambda \mathrm{a}\,.\,\lambda \mathrm{b}.\ \ \mathrm{a}\ \ \mathrm{b}\ \ \textbf{False}$$

$$\textbf{Or} = \lambda \mathrm{a}\,.\,\lambda \mathrm{b}.\ \ \mathrm{a}\ \ \textbf{True}\ \ \mathrm{b}$$

$$
\begin{aligned}
\textbf{Not}\ \ \textbf{True} &= (\lambda \mathrm{b}\,.\, b\ \ \textbf{False}\ \ \textbf{True})\ \ \textbf{True} \\
&\Rightarrow \textbf{True}\ \ \textbf{False}\ \ \textbf{True} \\
&= (\lambda \mathrm{t}\,.\, \lambda \mathrm{f}\,.\, t)\ \ \textbf{False}\ \ \textbf{True} \\
&\Rightarrow \textbf{False}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{And}\ \ \textbf{True}\ \ \textbf{True} &= (\lambda \mathrm{a}\,.\, \lambda \mathrm{b}\,.\, a\ \ b\ \ \textbf{False})\ \ \textbf{True}\ \ \textbf{True} \\
&\Rightarrow \textbf{True}\ \ \textbf{True}\ \ \textbf{False} \\
&= (\lambda \mathrm{t}\,.\, \lambda \mathrm{f}\,.\, t)\ \ \textbf{True}\ \ \textbf{False} \\
&\Rightarrow \textbf{True}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Or}\ \ \textbf{False}\ \ \textbf{True} &= (\lambda \mathrm{a}\,.\, \lambda \mathrm{b}\,.\, a\ \ \textbf{True}\ \ b)\ \ \textbf{False}\ \ \textbf{True} \\
&\Rightarrow \textbf{False}\ \ \textbf{True}\ \ \textbf{True} \\
&= (\lambda \mathrm{t}\,.\, \lambda \mathrm{f}\,.\, f)\ \ \textbf{True}\ \ \textbf{True} \\
&\Rightarrow \textbf{True}
\end{aligned}
$$

Chapter 3

Natural Numbers

$$\textbf{Zero} = \lambda \mathrm{s}\,.\,\lambda \mathrm{z}\,.\ \ \mathrm{z}$$

$$\textbf{Succ} = \lambda \mathrm{n}\,.\,\lambda \mathrm{s}\,.\,\lambda \mathrm{z}\,.\ \ \mathrm{s}\ \ (\mathrm{n}\ \ \mathrm{s}\ \ \mathrm{z})$$

$$\textbf{Pred} = \lambda \mathrm{n}\,.\,\lambda \mathrm{s}\,.\,\lambda \mathrm{z}\,.\ \ \mathrm{n}\ \ (\lambda \mathrm{f}\,.\,\lambda \mathrm{g}\,.\ \ \mathrm{g}\ \ (\mathrm{f}\ \ \mathrm{s}))\ \ (\lambda \mathrm{x}\,.\ \ z)\ \ \textbf{id}$$

$$\textbf{isZero} = \lambda \mathrm{n}\,.\ \ \mathrm{n}\ \ (\lambda \mathrm{x}\,.\ \ \textbf{False})\ \ \textbf{True}$$

$$\textbf{Plus} = \lambda \mathrm{m}\,.\,\lambda \mathrm{n}\,.\ \ \mathrm{m}\ \ \textbf{Succ}\ \ \mathrm{n}$$

$$\textbf{Minus} = \lambda \mathrm{m}\,.\,\lambda \mathrm{n}\,.\ \ \mathrm{m}\ \ \textbf{Pred}\ \ \mathrm{n}$$

$$\textbf{Mult} = \lambda \mathrm{m}\,.\,\lambda \mathrm{n}\,.\ \ \mathrm{m}\ \ (\textbf{Plus}\ \ \mathrm{n})\ \ \textbf{Zero}$$

$$\textbf{Power} = \lambda \mathrm{b}\,.\,\lambda \mathrm{e}\,.\ \ \mathrm{e}\ \ \mathrm{b}$$

$$
\begin{aligned}
\textbf{Succ}\ \ \textbf{Zero} &= (\lambda\mathrm{n}\,.\,\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ (n\ \ s\ \ z))\ \ \textbf{Zero}\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ (\textbf{Zero}\ \ s\ \ z)\\
&= \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ ((\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,z)\ \ s\ \ z)\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ z\\
&= \textbf{One}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Plus}\ \ \textbf{One}\ \ \textbf{One} &= (\lambda\mathrm{m}\,.\,\lambda\mathrm{n}\,.\,m\ \ \textbf{Succ}\ \ n)\ \ \textbf{One}\ \ \textbf{One}\\
&\Rightarrow \textbf{One}\ \ \textbf{Succ}\ \ \textbf{One}\\
&= (\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ z)\ \ \textbf{Succ}\ \ \textbf{One}\\
&\Rightarrow \textbf{Succ}\ \ \textbf{One}\\
&= (\lambda\mathrm{n}\,.\,\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ (n\ \ s\ \ z))\ \ \textbf{One}\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ (\textbf{One}\ \ s\ \ z)\\
&= \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ ((\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ z)\ \ s\ \ z)\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ (s\ \ z)\\
&= \textbf{Two}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Pred}\ \ \textbf{One} &= (\lambda\mathrm{n}\,.\,\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,n\ \ (\lambda\mathrm{f}\,.\,\lambda\mathrm{g}\,.\,g\ \ (f\ \ s))\ \ (\lambda\mathrm{x}\,.\,z)\ \ \textbf{id})\ \ \textbf{One}\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,\textbf{One}\ \ (\lambda\mathrm{f}\,.\,\lambda\mathrm{g}\,.\,g\ \ (f\ \ s))\ \ (\lambda\mathrm{x}\,.\,z)\ \ \textbf{id}\\
&= \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.(\lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,s\ \ z)\ \ (\lambda\mathrm{f}\,.\,\lambda\mathrm{g}\,.\,g\ \ (f\ \ s))\ \ (\lambda\mathrm{x}\,.\,z)\ \ \textbf{id}\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.(\lambda\mathrm{f}\,.\,\lambda\mathrm{g}\,.\,g\ \ (f\ \ s))\ \ (\lambda\mathrm{x}\,.\,z)\ \ \textbf{id}\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,\textbf{id}\ \ ((\lambda\mathrm{x}\,.\,z)\ \ s)\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.(\lambda\mathrm{x}\,.\,z)\ \ s\\
&\Rightarrow \lambda\mathrm{s}\,.\,\lambda\mathrm{z}\,.\,z\\
&= \textbf{Zero}
\end{aligned}
$$

Chapter 4

Pairs

$$\textbf{Pair} = \lambda \mathrm{a}\,.\,\lambda \mathrm{b}\,.\,\lambda \mathrm{p}\,.\;\; \mathrm{p\;\; a\;\; b}$$

$$\textbf{First} = \lambda \mathrm{p}\,.\;\; \mathrm{p}\;\; (\lambda \mathrm{a}\,.\,\lambda \mathrm{b}\,.\;\; \mathrm{a})$$

$$\textbf{Second} = \lambda \mathrm{p}\,.\;\; \mathrm{p}\;\; (\lambda \mathrm{a}\,.\,\lambda \mathrm{b}\,.\;\; \mathrm{b})$$

$$\textbf{Swap} = \lambda \mathrm{p}\,.\;\; \textbf{Pair}\;\; (\textbf{Second}\;\; \mathrm{p})\;\; (\textbf{First}\;\; \mathrm{p})$$

Nothing.

Chapter 5

Lists

$$\mathbf{Nil} = \lambda \mathrm{c}\,.\,\lambda \mathrm{n}\,.\;\; \mathrm{n}$$

$$\mathbf{Cons} = \lambda \mathrm{h}\,.\,\lambda \mathrm{t}\,.\,\lambda \mathrm{c}\,.\,\lambda \mathrm{n}\,.\;\; \mathrm{c}\;\; \mathrm{h}\;\; (\mathrm{t}\;\; \mathrm{c}\;\; \mathrm{n})$$

$$\mathbf{Append} = \lambda \mathrm{k}\,.\,\lambda \mathrm{l}\,.\;\; \mathrm{k}\;\; \mathbf{Cons}\;\; \mathrm{l}$$

$$\mathbf{ListSum} = \lambda \mathrm{l}\,.\;\; \mathrm{l}\;\; \mathbf{Plus}\;\; \mathbf{Zero}$$

Nothing.

Chapter 6

Binary Trees

$$\mathbf{Leaf} = \lambda \mathrm{v}\,.\,\lambda \mathrm{n}\,.\,\lambda \mathrm{l}\,.\;\; \mathrm{l}\;\; \mathrm{v}$$

$$\mathbf{Node} = \lambda \mathrm{s}\,.\,\lambda \mathrm{t}\,.\,\lambda \mathrm{n}\,.\,\lambda \mathrm{l}\,.\;\; \mathrm{n}\;\; (\mathrm{s}\;\; \mathrm{n}\;\; \mathrm{l})\;\; (\mathrm{t}\;\; \mathrm{n}\;\; \mathrm{l})$$

$$\mathbf{TreeSum} = \lambda \mathrm{t}\,.\;\; \mathrm{t}\;\; \mathbf{Plus}\;\; \mathbf{id}$$

Nothing.

Chapter 7

Recursion

$$\mathbf{Y} = \lambda\mathrm{f}.(\lambda\mathrm{x}.\, f\ (x\ \ x))\ (\lambda\mathrm{x}.\, f\ (x\ \ x))$$

$$
\begin{aligned}
\mathbf{Y}\ \ f &= (\lambda\mathrm{f}\,.(\lambda\mathrm{x}\,.\,f\ \ (x\ \ x))\ \ (\lambda\mathrm{x}\,.\,f\ \ (x\ \ x)))\ \ f \\
&\Rightarrow (\lambda\mathrm{x}\,.\,f\ \ (x\ \ x))\ \ (\lambda\mathrm{x}\,.\,f\ \ (x\ \ x)) \\
&\Rightarrow f\ \ ((\lambda\mathrm{x}\,.\,f\ \ (x\ \ x))\ \ (\lambda\mathrm{x}\,.\,f\ \ (x\ \ x))) \\
&= f\ \ (\mathbf{Y}\ \ f) \\
&\Rightarrow \ldots
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{fact} &= \lambda\mathrm{f}\,.\,\lambda\mathrm{n}\,.(\mathbf{isZero}\ \ n)\ \ \mathbf{One}\ (\mathbf{Mult}\ \ n\ (f\ \ (\mathbf{Pred}\ \ n))) \\
\mathbf{Fact} &= \mathbf{Y}\ \ \mathbf{fact}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Fact}\ \ \mathbf{Two} &= \mathbf{Y}\ \ \mathbf{fact}\ \ \mathbf{Two} \\
&\Rightarrow \mathbf{fact}\ \ (\mathbf{Y}\ \ \mathbf{fact})\ \ \mathbf{Two} \\
&= (\lambda\mathrm{f}\,.\,\lambda\mathrm{n}\,.(\mathbf{isZero}\ \ n)\ \ \mathbf{One}\ (\mathbf{Mult}\ \ n\ (f\ \ (\mathbf{Pred}\ \ n))))\ (\mathbf{Y}\ \ \mathbf{fact})\ \ \mathbf{Two} \\
&\Rightarrow (\mathbf{isZero}\ \ \mathbf{Two})\ \ \mathbf{One}\ (\mathbf{Mult}\ \ \mathbf{Two}\ ((\mathbf{Y}\ \ \mathbf{fact})\ (\mathbf{Pred}\ \ \mathbf{Two}))) \\
&\Rightarrow \mathbf{False}\ \ \mathbf{One}\ (\mathbf{Mult}\ \ \mathbf{Two}\ ((\mathbf{Y}\ \ \mathbf{fact})\ \ \mathbf{One})) \\
&\Rightarrow \mathbf{Mult}\ \ \mathbf{Two}\ ((\mathbf{Y}\ \ \mathbf{fact})\ \ \mathbf{One}) \\
&\Rightarrow \mathbf{Mult}\ \ \mathbf{Two}\ ((\mathbf{fact}\ \ (\mathbf{Y}\ \ \mathbf{fact}))\ \ \mathbf{One}) \\
&= \mathbf{Mult}\ \ \mathbf{Two}\ ((\lambda\mathrm{f}\,.\,\lambda\mathrm{n}\,.(\mathbf{isZero}\ \ n)\ \ \mathbf{One}\ (\mathbf{Mult}\ \ n\ (f\ \ (\mathbf{Pred}\ \ n)))\ (\mathbf{Y}\ \ \mathbf{fact}))\ \ \mathbf{One}) \\
&\Rightarrow \mathbf{Mult}\ \ \mathbf{Two}\ (((\mathbf{isZero}\ \ \mathbf{One})\ \ \mathbf{One}\ (\mathbf{Mult}\ \ \mathbf{One}\ ((\mathbf{Y}\ \ \mathbf{fact})\ (\mathbf{Pred}\ \ \mathbf{One})))\ )\ )
\end{aligned}
$$